'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 6th Semester Examination, 2023

CC13-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-II

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

Answer any *four* questions from the following $3\times 4=12$

1. Show that the polynomial $3x^5+15x^4-20x^3+10x+20$ is irreducible over $\mathbb{Q}$.

2. Suppose that a, b are two elements in an integral domain, $b\neq 0$ and a is not a unit. Show that $\langle ab\rangle$ is contained in $\langle b\rangle$.

3. Let $V=C[0,1]$ and define $\langle f, g\rangle=\int_0^{1/2} f(t)g(t)\,dt$. Is this an inner product on V ?

4. Prove that the ideal $\langle x^2+1\rangle$ is prime in $\mathbb{Z}[x]$ but not maximal in $\mathbb{Z}[x]$.

5. Let $V=P_1(\mathbb{R})$ and for $p(x)\in V$, define $f_1, f_2\in V^*$, by $f_1(p(x))=\int_0^1 p(t)\,dt$ and $f_2(p(x))=\int_0^2 p(t)\,dt$. Prove that $\{f_1, f_2\}$ is a basis for V^*.

6. Prove that $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ is not diagonalizable.

GROUP-B

Answer any *four* questions from the following $6\times 4=24$

7. (a) Prove that every Euclidean Domain is a PID. 4

   (b) Prove that in an integral domain, associates of every irreducible element are also irreducible. 2

8. (a) Let T be a linear operator on a finite dimensional vector space V and let W be a T-invariant subspace of V. Then prove that the characteristic polynomial of T_W divides the characteristic polynomial of T. 3

(b) Let V be an inner product space and S_1 and S_2 be two subsets of V. Then prove that $S_1 \subseteq S_2 \Rightarrow S_1^{\perp} \subseteq S_2^{\perp}$. 3

9. (a) Let F be a field and $p(x) \in F[x]$. Then prove that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ iff $p(x)$ is irreducible over F. 4

(b) Prove that 2 and 5 are not irreducible elements in $\mathbb{Z}[i]$. 2

10.(a) Prove that in an integral domain, two elements a and b are associates iff $\langle a \rangle = \langle b \rangle$. 3

(b) Show that $1+2i$ and $3+5i$ are prime to each other in $\mathbb{Z}[i]$. 3

11.(a) In $\mathbb{R}^3$, with standard inner product, let P be the subspace $\text{span}\{(1, 1, 0), (0, 1, 1)\}$. Find $P^{\perp}$. 3

(b) Let V be a finite dimensional Euclidean space. Then prove that a linear mapping $T : V \to V$ is orthogonal iff T maps an orthonormal basis to another orthonormal basis. 3

12.(a) Prove that the set of all normal operators is a closed subset of $L(H, H)$ which contains the set of all self-disjoint operators. 4

(b) Suppose $A \in L(H, H)$. Then prove that $\langle Ax, x \rangle = 0$ for all $x \in H$ iff $A = 0$. 2

GROUP-C

Answer any *two* questions from the following $12 \times 2 = 24$

13.(a) Let $A : H \to H$ is a continuous linear operator, where H is a Hilbert space. Prove that A^* is a continuous linear operator with $\| A^* \| = \| A \|$. 5

(b) Prove that the dual space of an n dimensional vector space is n dimensional. 4

(c) Find the minimal polynomial of the matrix 3

$$A = \begin{pmatrix} -3 & 2 \\ 0 & -3 \end{pmatrix}$$

14.(a) Let T be an linear operator on $V = \mathbb{R}^2$ defined by $T(a, b) = (2a - 2b, -2a + 5b)$ for all $(a, b) \in \mathbb{R}^2$. Determine whether T is normal, self-adjoint or neither. Produce an orthonormal basis of eigenvectors of T for V. 6

(b) Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard ordered bases β and γ. Define $T : V \to W$ by $T(p(x)) = (p(0) - 2p(1), p(0) + p'(0))$; where $p'(x)$ denotes the derivative of $p(x)$. Then compute $[T^t]_{\gamma^*}^{\beta^*}$ and $[T]_{\beta}^{\gamma}$. 6

15.(a) Apply Gram-Schmidt process to the subset $S = \{1, x, x^2\}$ of the inner product space $V = P_2(\mathbb{R})$ with inner product 4

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$$

to obtain an orthonormal basis β for $\text{span}(S)$.

(b) For two subspaces W_1 and W_2 of a finite dimensional vector space V, prove that 3

$(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(c) Let T be a linear operator on a finite dimensional vector space V and let $f(t)$ be the characteristic polynomial of T. Then prove that $f(T) = T_0$, the zero transformation. 5

16.(a) Show that the following polynomials are irreducible: 6

(i) $x^3 - [9]$ over $\mathbb{Z}_{11}$.

(ii) $x^4 + 2x + 2$ over $\mathbb{Q}$.

(iii) $x^6 + x^3 + 1$ over $\mathbb{Q}$.

(b) Let R be the ring $\mathbb{Z} \times \mathbb{Z}$. Show that the linear equation $(5, 0)x + (20, 0) = (0, 0)$ has infinitely many roots in R. 3

(c) In $\mathbb{Z}_7[x]$, factorize $f(x) = x^3 + [1]$ into linear factors. 3

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